

ON METRIC GRAPHS WITH PRESCRIBED GONALITY

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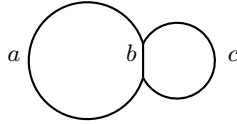
ABSTRACT. We prove that in the moduli space of genus- g metric graphs the locus of graphs with gonality at most d has the classical dimension

$$\min\{3g - 3, 2g + 2d - 5\}.$$

This follows from a careful parameter count to establish the upper bound and a construction of sufficiently many graphs with gonality at most d to establish the lower bound. Here, gonality is the minimal degree of a non-degenerate harmonic map to a tree that satisfies the Riemann-Hurwitz condition everywhere. Along the way, we establish a convenient combinatorial datum capturing such harmonic maps to trees.

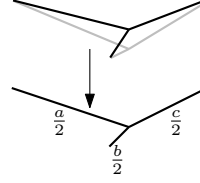
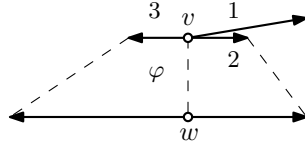
1. DEFINITIONS AND RESULT

Metric graphs. A *topological graph* is a topological space obtained by gluing a finite, disjoint union of closed intervals along an equivalence relation on the end-points. If a topological graph Γ is connected and the intervals from which it is glued are prescribed with a positive length, then Γ becomes a compact metric space with shortest-path metric. Such a metric space is called a *metric graph*. The *genus* of a metric graph is its cycle space dimension. Here is a metric graph of genus 2 with edge lengths a, b, c :



Every point v in a metric graph has a neighbourhood isometric, for some positive ϵ , to a finite union of half-open intervals $[0, \epsilon)$ glued along 0. Each of these intervals is called a *half-edge* emanating from v , and their number is the *valency* of v . We identify two half-edges emanating from v (for different ϵ) if one is contained in the other.

Harmonic maps. A map φ from a metric graph Γ to a metric graph Σ is called *harmonic* if it is continuous, linear with integral slopes outside a finite number of points, and if it moreover satisfies the following harmonicity condition at each point $v \in \Gamma$: Fix a half-edge e emanating from $w := \varphi(v)$, and consider the sum of all slopes of φ along half-edges emanating from v that map to e . That sum, denoted $m_\varphi(v)$, should be independent of the choice of e . A harmonic map has a well-defined degree, defined as $\deg \varphi := \sum_{v \in \Gamma, \varphi(v)=w} m_\varphi(v)$ for any $w \in \Sigma$. On the left is an example with $m_\varphi(v) = 3$, and on the right is an example of a degree-2 harmonic map from our earlier genus-2 graph to a metric tree:



This definition is closely related to the notion of *pseudo-harmonic indexed morphisms* in [Cap14] (which generalise harmonic morphisms from [BN09]) as follows: there the set-up concerns ordinary (non-metric) graphs, and the role of the index there is played by the slope in our definition. Moreover, in [Cap14] the definition is extended to graphs in which the vertices are decorated with a non-negative genus; we do not do so here.

Riemann-Hurwitz conditions and tropical morphisms. Let $\varphi : \Gamma \rightarrow \Sigma$ be a harmonic map and $v \in \Gamma$. The *Riemann-Hurwitz condition* [BBM11, Definition 2.2] on φ at v is the inequality

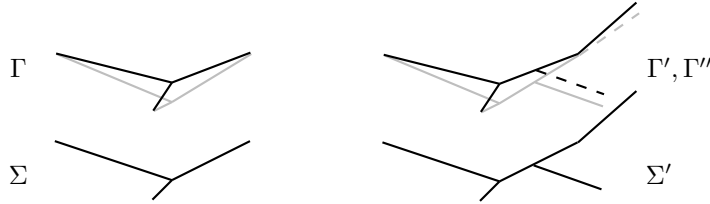
$$k - 2 \geq m_\varphi(v) \cdot (l - 2),$$

where k is the valency of v and l the valency of $w := \varphi(v)$. We will only be interested in harmonic maps that satisfy the Riemann-Hurwitz condition at all $v \in \Gamma$ and that, moreover, have *non-zero* integral slopes outside a finite number of points. This latter condition is equivalent to saying that $m_\varphi(v)$ is strictly positive at all $v \in \Gamma$. This is related to the condition of *non-degeneracy* in [Cap14] as follows: there the non-degeneracy is required at vertices, and not at “internal points” of edges. But by folding each edge contracted to a point into two, one can create a harmonic morphism between corresponding metric graphs with non-zero slopes, albeit that the target graph needs to be modified as discussed below. The non-zero slope condition is also related to the modifiability condition in [BBM14, Remark 2.5] and the finiteness condition in [ABBR14, Definition 2.4].

We will call harmonic maps satisfying the Riemann-Hurwitz condition and the non-zero slope condition everywhere *tropical morphisms* between metric graphs.

Modifications. A *modification* of a metric graph Γ is any metric graph Γ' obtained from Γ by grafting a finite number of metric trees onto points of Γ . Given a modification Γ' of Γ and a tropical morphism $\varphi : \Gamma \rightarrow \Sigma$, there exist a modification Γ'' of Γ' , a modification Σ' of Σ , and a tropical morphism map $\varphi' : \Gamma'' \rightarrow \Sigma'$ of the same degree as φ and extending φ , constructed as follows. For any tree S grafted onto $v \in \Gamma$ when going from Γ to Γ' , Γ'' has $m_\varphi(v) - 1$ additional copies of S grafted onto v , and Σ' has a single copy of S grafted onto $\varphi(v)$. Here we use that $m_\varphi(v)$ is positive, which follows from the fact that slopes are non-zero. The map φ' equals φ on Γ and maps the $m_\varphi(v)$ copies of S in Γ'' to the single copy in Σ' .

On the right is a modification of the harmonic morphism on the left:



Here the two solid segments are grafted onto Γ to arrive at Γ' , and to arrive at Γ'' the two additional dashed lines are grafted.

Gonality. The *tree gonality* of a metric graph Γ is the minimum degree of any tropical morphism from any modification Γ' of Γ to a tree.

There are two notions of graph gonality in the literature, which are both inspired by the gonality of an algebraic curve. They are tree (or *geometric*) gonality and *divisorial* gonality. Each of these comes in several flavours, e.g. for ordinary or metric graphs [Bak08], for graphs where the vertices can be decorated with higher genera [Cap14], and for metrized complexes [AB15, LM14]. Yet another variant is *stable gonality*, which is the minimum of the divisorial gonality over all subdivisions of an ordinary graph [CK13]. In this paper, *gonality* will always refer to tree gonality of metric graphs as defined above.

Given such a tropical morphism $\varphi : \Gamma' \rightarrow T$ of degree d , and any $w \in T$, the divisor $D' := \sum_{v \in \varphi^{-1}(w)} m_\varphi(v) \cdot (v)$ on Γ' has degree d and rank [BN07] at least one. Moving the chips of D on grafted trees to their grafting points, one obtains a divisor D on Γ of degree d and rank at least one. In particular, tree gonality is at least divisorial gonality. Divisorial gonality, in turn, is bounded from below by the treewidth of a graph or metric graph [vDdBG14, AK14].

The moduli space of metric graphs. Fix a natural number $g \geq 2$. For any genus- g graph $G = (V, E)$ (that is, a connected graph with $|E| - |V| + 1 = g$; multiple edges and loops are allowed) with all valencies greater than or equal to 3 we have $|E| \leq 3g - 3$, with equality if and only if all valencies are 3. The positive orthant $M_G = \mathbb{R}_{>0}^E$ of edge lengths parameterises metric graphs of *combinatorial type* G . If H is obtained from G by contracting an edge e (not a loop), then H is another genus- g graph, and we glue M_H to M_G as the coordinate hyperplane $\mathbb{R}_{>0}^{E \setminus \{e\}} \times \{0\}$. Similarly, if $H = (V', E')$ is a graph isomorphic to G via an isomorphism $\sigma : E \rightarrow E'$, then we glue M_H to M_G via the induced bijection $\mathbb{R}_{>0}^{E'} \rightarrow \mathbb{R}_{>0}^E$. In particular, we do this for automorphisms of G . This gluing of all M_G where G runs over all genus- g graphs yields a topological space known as $\mathcal{M}_g^{\text{trop}}$, the *moduli space of metric graphs of genus g* . It has dimension $3g - 3$. By construction, every genus- g graph $G = (V, E)$ defines a map $\mathbb{R}_{>0}^E \rightarrow \mathcal{M}_g^{\text{trop}}$. Every genus- g metric graph is a modification of some graph represented by a point in $\mathcal{M}_g^{\text{trop}}$. The generalisation of $\mathcal{M}_g^{\text{trop}}$ to graphs with weighted vertices and marked points is discussed, for instance, in [Cap12, Section 3], and the topology of these spaces is the topic of [Koz09].

Main results. We will prove the following two theorems.

Theorem 1. *For $d, g \geq 2$ the locus of metric graphs in $\mathcal{M}_g^{\text{trop}}$ that have gonality at most d is closed of dimension $\min\{2g + 2d - 5, 3g - 3\}$. In particular, the locus of genus- g metric graphs of gonality at least $\lceil (g + 2)/2 \rceil$ is open and dense in $\mathcal{M}_g^{\text{trop}}$.*

This theorem comprises two inequalities, and both will be proved by purely combinatorial means. Using the Kempf-Kleiman-Laksov existence result for special divisors [Kem71, KL72] and [Cap14, Theorem 2.11] (which is a variant of Baker's specialisation lemma from [Bak08]), it follows that in fact *all* metric graphs in $\mathcal{M}_g^{\text{trop}}$ have gonality at most $\lceil (g + 2)/2 \rceil$. We will give a constructive, purely combinatorial proof for the following statement.

Theorem 2. *Let $g \geq 1$ be a natural number. For any genus- g graph $G = (V, E)$ all of whose vertices have valency 3, the positive orthant $(\mathbb{R}_{>0})^E$ contains a non-empty open cone C_G whose image in $\mathcal{M}_g^{\text{trop}}$ consists entirely of metric graphs with gonality exactly $d := \lceil (g+2)/2 \rceil$. Moreover, C_G can be chosen such that every metric graph represented by a point in $C_G \cap \mathbb{Z}_{>0}^E$ has a degree- d divisor of rank 1 supported at integral points.*

This theorem implies the upper bound on the *stable gonality* from [CK13, Theorem B], which is defined by taking the infimum of the tree gonality over all subdivisions of an ordinary graph. It has been conjectured that in fact *all* metric graphs with integral edge lengths admit such a divisor [Bak08, Conjecture 3.10], but our methods do not imply that result. We further remark that by [Luo11], the rank of a divisor on a *metric* graph supported on integral points equals its rank when regarded as a divisor on the natural *ordinary graph* whose vertices are the integral points.

The fact that the locus of metric graphs of gonality (at least) $\lceil (g+2)/2 \rceil$ is open and dense in $\mathcal{M}_g^{\text{trop}}$ is an exact tropical analogue of the corresponding statement for algebraic curves. This is interesting as, so far, only the (divisorial) gonality of rather specific graphs (such as chains of loops) was well-understood [CDPR12]. Using the aforementioned variant due to Caporaso of Baker’s specialisation lemma, our theorem implies that a general curve of genus g has gonality at least $\lceil (g+2)/2 \rceil$ (the “non-existence part” of Brill-Noether theory). The idea that one does not need a *specific* graph to prove this statement but that, rather, a suitable dimension count suffices, goes back to a conference talk by Mikhalkin in 2011 [Mik11].

Our paper is organised as follows. In Section 2 we present a combinatorial datum that captures a metric graph together with a tropical morphism of degree d to a tree. We call this a *gluing datum*. Using this datum, in Section 3 we prove the upper bound on the dimension of the locus of gonality- d metric graphs in $\mathcal{M}_g^{\text{trop}}$. Finally, in Section 4 we construct the cone C_G and variants of it for lower-than-maximal gonality, thus showing that the upper bound is in fact the right dimension.

The most important problem that we leave open is to find a combinatorial construction of a degree- $\lceil (g+2)/2 \rceil$ tropical morphism φ from (a modification of) *every* metric graph Γ of genus g to a tree. Moreover, since our construction of such morphisms for graphs in the cone C_G depends continuously on the edge lengths, and since the moduli space of metric graphs of genus g is connected in codimension one by [Cap12], it is natural to ask whether the space of such pairs (Γ, φ) is in a suitable sense connected.

2. THE GLUING DATUM

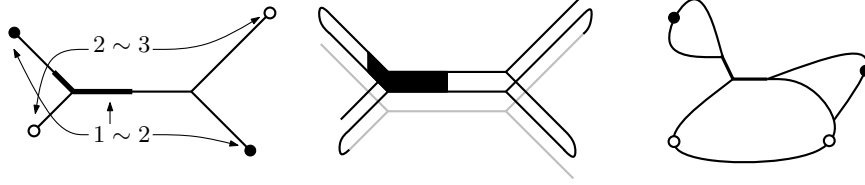
We define the following combinatorial gadget.

Definition 3. A *gluing datum* (T, d, \sim) is a tuple consisting of a metric tree T , a natural number d , and an equivalence relation \sim on the disjoint union $S := T_1 \sqcup \dots \sqcup T_d$ of d copies of T satisfying the following properties, in which $\psi_i : T \rightarrow T_i$ stands for the identification of T with its i -th copy.

- (1) If $v, w \in S$ satisfy $v \sim w$, then there exist $u \in T$ and $i, j \in [d]$ such that $\psi_i(u) = v \in T_i$ and $\psi_j(u) = w \in T_j$.

- (2) For any pair $i \neq j$ the set $\{u \in T \mid \psi_i(u) \sim \psi_j(u)\}$ has finitely many connected components, each of which is either a single point or homeomorphic to a closed interval.
- (3) The topological graph Γ obtained from S by identifying points along the equivalence relation \sim is connected.

Here is an example with $d = 3$ leading to a genus-3 graph; the colors are purely decorative. In the last picture, dangling trees have been removed.



It is convenient to think of the d copies of T as lying above each other. Then the first condition says that a point is glued only to points vertically above or below it. The second condition is the trickiest one; it says that the point set along which T_i and T_j are glued is closed, has finitely many connected components, and that each component is either a closed interval or a point (which we consider a closed interval of length zero). For instance, gluing two components along a tripod is not allowed. This condition will become natural after two lemmas below.

We turn the topological graph Γ into a metric graph as follows: if $v \in \Gamma$ is not an endpoint of any of the gluing intervals (of positive or zero length), then locally near v a fixed number d_v of the T_i were glued together. Near v we give Γ the metric of T divided by d_v , so that the natural map $\Gamma \rightarrow T$ has slope equal to d_v near v .

A gluing datum and a point $w \in T$ together give rise to an equivalence relation \sim_w on the set $[d]$ expressing which copies of T are glued at w . By the closedness of the gluing point sets, the map $w \mapsto \sim_w$ is semicontinuous in the following sense: if $w_i \rightarrow w$ for $i \rightarrow \infty$, then for sufficiently large i the equivalence relation \sim_w is a coarsening of each of the equivalence relations \sim_{w_i} .

Proposition 4. *Let (T, d, \sim) be a gluing datum, and let $\Gamma = S/\sim$ be the topological space obtained from it, equipped with the aforementioned metric. Then the natural map $\varphi : \Gamma \rightarrow T$ is a tropical morphism of degree d .*

Proof. The map φ is continuous; this follows from the “universal property” of the quotient map $S \rightarrow \Gamma = S/\sim$, namely, that any map from S into a topological space such that points equivalent under \sim are mapped to the same point (here the map $S \rightarrow T$) factorises through a continuous map from Γ (here φ). The slope of φ at any point v that is not an endpoint of a gluing interval is $d_v > 0$.

To see that φ is harmonic, let $v \in \Gamma$ and let e be a half-edge emanating from $w := \varphi(v)$ in T . Then the points near (but unequal to) w along e induce a fixed equivalence relation \sim_e on $[d]$, which refines the equivalence relation \sim_w on $[d]$. Let $I \subseteq [d]$ be the equivalence class of \sim_w containing v , which decomposes as a disjoint union $I_1 \sqcup \dots \sqcup I_k$ of equivalence classes under \sim_e . The latter classes correspond bijectively to the half-edges of Γ emanating from v that map to e . On the half-edge corresponding to I_j , the metric has been defined such that the slope of φ equals $|I_j|$. Adding up all these slopes yields $m_\varphi(v) := |I|$, which is an invariant of v and in particular independent of the half-edge e . This proves harmonicity. The degree is d because that is the sum of the cardinalities of all equivalence classes of \sim_w .

Finally, we need to argue that the Riemann-Hurwitz conditions are satisfied. Consider a point $v \in \Gamma$ coming from the i -th copy T_i . Let k be the valency of v and l be the valency of $w := \varphi(v) \in T$. The third quantity in the Riemann-Hurwitz equation is the number $m_\varphi(v)$, which is the number of T_j glued to T_i at v , including T_i itself. Let T_j with $j \neq i$ be one of these, and let e_j be the number of half-edges emanating from v along which T_j is glued to T_i . Then $e_j \leq 2$ since T_i and T_j are glued only along intervals. Hence we find

$$k = m_\varphi(v)l - \sum_{j \sim_w i, j \neq i} e_j$$

where the sum is over all T_j with $j \neq i$ that are glued to T_i at the point v . Hence the Riemann-Hurwitz inequality $(k - 2) \geq m_\varphi(v)(l - 2)$ is equivalent to

$$\sum_{j \sim_w i, j \neq i} e_j + 2 \leq 2m_\varphi(v)$$

Since the number of j with $j \sim_w i \neq j$ is equal to $m_\varphi(v) - 1$ and since each e_j is at most 2, the left-hand side is at most $2(m_\varphi(v) - 1) + 2 = 2m_\varphi(v)$, as required. \square

As we will prove next, every tropical morphism to a tree arises from a suitable gluing datum. This datum is not unique. First, there is the obvious ambiguity arising from permuting the copies T_1, \dots, T_d . But in fact there is much more ambiguity, as will become apparent in the following proof.

Proposition 5. *Let $\varphi : \Gamma \rightarrow T$ be a tropical morphism. Then there exists a gluing datum (T, d, \sim) that gives rise to φ .*

Proof. The number d is defined as the degree of φ . Without loss of generality, we may assume that the pre-image of some leaf w_0 of T consists of d distinct leaves v_1, \dots, v_d of Γ with $m_\varphi(v_i) = 1$ for all i . This situation can be achieved by grafting an additional interval on some leaf of T and extending the morphism as discussed in the section on modifications.

We now construct the gluing relation \sim on the disjoint union $T_1 \sqcup \dots \sqcup T_d$ of T , by determining the local relation \sim_w at points $w \in T$ as follows. Start at the leaf $w_0 \in T$, and let $\{v_1, \dots, v_d\} \subseteq \Gamma$ be the pre-image of w_0 under φ . For the gluing relation \sim_{w_0} at w_0 choose the partition Π_0 of $[d]$ into singletons.

Initialise $w := w_0$ and $\Pi := \Pi_0$, and start moving w along T , tracing the pre-image $\{v_1, \dots, v_d\}$ along. When you arrive at a point w of T where the partition Π just before w equals Π_1 , the following happens: some of the v_i (perhaps none) will converge, and we are forced to mimick this in the gluing datum by making Π_w the corresponding coarsening of Π_1 . Let e_1, \dots, e_l be the half-edges emanating from w in T , where e_1 is in the direction of w_0 . Along the remaining half-edges e_2, \dots, e_l emanating from w (so all except for e_1), we still have to choose the partitions Π_2, \dots, Π_l of $[d]$. They should satisfy the following rules. First, each Π_i should be a refinement of Π_w . Second, the number of parts of Π_i into which the part of Π_w corresponding to a $v \in \varphi^{-1}(w)$ splits equals the number of half-edges in Γ emanating from v that map to e_i . Third, the cardinalities of these parts should equal the slopes of φ along those half-edges. As φ is harmonic, these slopes add up to the same number $m_\varphi(v)$ for all i , so that such partitions Π_2, \dots, Π_l certainly exist.

However, we still need to ensure that two trees T_i, T_j are glued along intervals only. This imposes the following combinatorial condition on the partitions

Π_1, \dots, Π_l : if i, j are distinct elements in the same part of some partition Π_h , then there is at most one $h' \neq h$ such that i, j are in the same part of $\Pi_{h'}$. Lemma 6 below shows that the Riemann-Hurwitz condition implies that Π_2, \dots, Π_l can be chosen to ensure this. \square

Lemma 6. *Let l, m, k_1, \dots, k_l be positive integers and assume that $k_1 + \dots + k_l - 2 \geq m(l - 2)$. For each $h = 1, \dots, l$ let π_h be a partition of the number m with k_h (nonzero) parts; in particular, each k_h is at most m .*

Then there exist partitions Π_1, \dots, Π_l of the set $[m]$ such that π_h records the sizes of the parts in Π_h and such that the coarsest common refinement of any three of the Π_h is the partition into singletons.

We dub this property of a sequence of partitions the *triple intersection property*.

Proof. We proceed by induction on m . The statement is trivially true for $m = 1$: the only choice for each Π_h is the partition of $[1]$ into the one singleton $\{1\}$, and this choice satisfies the triple intersection property. Now let $m \geq 2$, assume that the statement is true for $m - 1$, and consider k_1, \dots, k_l and partitions π_h of m as in the lemma. Order the k_h such that $k_1 \leq \dots \leq k_l$.

If $k_2 = m$, then the only choices for Π_2, \dots, Π_l are the partitions into singletons, and any choice for Π_1 will do. Hence we may assume that $k_1, k_2 < m$, so that π_1, π_2 contain parts $a_1, a_2 > 1$, respectively. Next, we have $k_3 > 1$ because otherwise

$$\sum_h k_h - 2 \leq 3 \cdot 1 + (l - 3) \cdot m - 2 < (l - 2)m,$$

where we use that $m > 1$. Hence π_3 contains at least two parts $a_3, b_3 > 0$. Similarly, we find that $k_4 > m/2$, hence each π_h with $h \geq 4$ contains at least one 1.

Now construct π'_h from π_h as follows: for $h = 1, 2$ reduce the part a_h by 1; for $h = 3$ replace the two parts a_3, b_3 by a single part $a_3 + b_3 - 1$; and for $h \geq 4$ discard a part equal to 1. This yields l partitions of $m - 1$, and in the inequality of the lemma both sides have been reduced by $l - 2$. By the induction hypothesis, there are partitions Π'_h , $h = 1, \dots, l$ of $[m - 1]$ corresponding to π'_h that satisfy the triple intersection property.

From the Π'_h we construct partitions Π_h of $[m]$ as follows. For $h = 1, 2$ let A'_h be the (or a) part of Π'_h of cardinality $a_h - 1$; add m to this set to obtain A_h . For $h > 3$ add the singleton $\{m\}$ to the partition Π'_h . Finally, for $h = 3$ let A'_3 be the part of Π'_3 of size $a_3 + b_3 - 1$. We want to replace A'_3 with two sets B_3 and $A_3 := (A'_3 \setminus B_3) \cup \{m\}$ where $B_3 \subseteq A'_3$ has cardinality b_3 . The only triple intersection that might now get cardinality 2 is the one between A_1, A_2, A_3 (which contains m), but this happens only if we put an element of $A'_1 \cap A'_2 \cap A'_3$ in A_3 . Since this intersection contains at most one element, we can avoid this by putting that element, if it exists, into B_3 (whose prescribed cardinality b_3 is positive). \square

Remark 7. We think that a generalisation of Lemma 6 might hold, where one replaces 2 by an $n \in \{0, \dots, l - 1\}$, the inequality by $k_1 + \dots + k_l - n \geq m(l - n)$, and the triple intersection property by the property that the coarsest common refinement of any $n + 1$ of the partitions be the partition into singletons. But since we do not need this for our current purposes, we have not pursued this.

Now that we know that gluing data give rise to tropical morphisms and vice versa, we can express the genus of a metric graph Γ in terms of a gluing datum as follows.

Proposition 8. *Let (T, d, \sim) be a gluing datum. For each subset $I \subseteq [d]$ define*

$$T_I := \{w \in T \mid \forall i, j \in I : i \sim_w j\} \subseteq T.$$

Then the genus of the metric graph Γ determined by the datum equals

$$g(\Gamma) = \sum_{I \subseteq [d]} (-1)^{|I|} c(T_I),$$

where $c(T_I)$ is the number of connected components of T_I .

Before we proceed with the proof, observe that

$$T_I = \bigcap_{i, j \in I, i \neq j} \{w \in T \mid i \sim_w j\},$$

where the intersection is considered all of T if $|I| < 2$. Each of the sets in this intersection is a union of disjoint closed intervals (possibly of length zero), hence so is the intersection; the number $c(T_I)$ counts these intervals. In the proof, however, we will not need that T_i, T_j are glued only along intervals.

Proof. Since Γ is connected by assumption, we may order the copies of T such that each T_i with $i > 1$ is glued to at least one T_j with $j < i$. For each $e \in [d]$ the restriction of \sim to $T_1 \sqcup \dots \sqcup T_e$ then yields a gluing datum of a connected metric graph Γ_e , which is obtained from Γ_{e-1} by suitably gluing the copy T_e to it.

We argue by induction on e . For $e = 1$ we observe that both $c(T_\emptyset)$ and $c(T_{\{1\}})$ are 1, and hence the alternating sum is $1 - 1 = 0$, which is the genus of the tree $\Gamma_1 = T_1$. Now assume that the formula holds for $e - 1$, and consider the graph Γ_e . Let $C := \{w \in T \mid \exists i < e : i \sim_w e\}$ be the set along which T_e is glued to Γ_{e-1} . This is some closed forest in T with finitely many connected components. Gluing Γ_e and T_e along the first connected component of C yields a graph equal to Γ_{e-1} but with some trees grafted onto it, hence of the same genus as Γ_{e-1} . Gluing along every further component of C increases the genus by 1. Thus the genus of Γ_e equals that of Γ_{e-1} plus the number of connected components of C minus 1. Comparing to the formula in the lemma, we need to argue that the number of connected components of C equals

$$\sum_{I \subseteq [e-1], |I| > 0} (-1)^{(|I|+1)} c(T_{I \cup \{e\}}).$$

But this follows from the equality $C = \bigcup_{i=1}^{e-1} T_{i,e}$ and the fact that closed forests X, Y in T with finitely many connected components satisfy $c(X \cup Y) = c(X) + c(Y) - c(X \cap Y)$ —for instance, because in this case c equals the Euler characteristic. \square

3. UPPER BOUNDS ON THE DIMENSION OF THE LOCUS OF BOUNDED GONALITY

Our goal in this section is to derive the upper bound from Theorem 1 on the dimension of the locus in $\mathcal{M}_g^{\text{trop}}$ where the gonality is equal to d . For a gluing datum (T, d, \sim) we call a point $w \in T$ an *endpoint* if it is an endpoint of some connected component of $T_{\{i,j\}}$ for some distinct $i, j \in [d]$; and we write E for the set of endpoints that have valency at most 2 in T . We also introduce the following notation: if Γ is the corresponding metric graph and $\varphi : \Gamma \rightarrow T$ the tropical morphism defined by the datum and $v \in \Gamma$ a point with valency k where $\varphi(v)$ has valency l in T , then we set $r_\varphi(v) := (k-2) - m_\varphi(v)(l-2)$. By the Riemann-Hurwitz condition, this is a nonnegative number. Note that $r_\varphi(v)$ is positive only at a finite number of points.

Proposition 9. *Let (T, d, \sim) be a gluing datum, $\varphi : \Gamma \rightarrow T$ the corresponding tropical morphism, and g the genus of Γ . Then we have*

$$(1) \quad |E| + \sum_{v \in T \text{ of valency } > 2} r_\varphi(v) \leq 2g + 2d - 2.$$

Proof. We proceed by induction as in the proof of Proposition 8, and use the notation of that proof. For $d = 1$ the statement is true: $E = \emptyset$, $r_\varphi(v) = 0$ for all v , and $0 + 0 = 2 \cdot 0 + 2 \cdot 1 - 2$. For the induction step, let C be the set along which T_e is glued to Γ_{e-1} . The increase in g when passing from Γ_{e-1} to Γ_e equals $c(C) - 1$, and the increase in d is 1. Hence the right-hand side in (1) increases with $2c(C)$.

On the other hand, consider a connected component C' of C and let $v \in C'$. Denote by l_v the valency of v in T and by l'_v the valency of v in the closed tree C' . Then $m_\varphi(v)$ increases by 1 in passing from Γ_{e-1} to Γ_e , while the valency k of v in these graphs increases by $l_v - l'_v$. This means that $r_\varphi(v)$ increases by $l_v - l'_v - (l_v - 2) = 2 - l'_v$ (so a decrease if $l'_v > 2$).

If C' consists of the single point v only, then v either contributes to an increase of $|E|$ by at most 1 (if $l_v \leq 2$) or to an increase by $2 - l'_v = 2$ of the second summand in the left-hand side of (1) (if $l_v > 2$). In either case, C' contributes an increase of at most 2 to the left-hand side.

Now suppose that C' is not a single point. Then each boundary point v of C' has $l'_v = 1$ and contributes at most 1 to the left-hand side: to $|E|$ if $l_v \leq 2$ or to the other summand if $l_v > 2$. An interior point v of C' cannot be a new point in E . Indeed, if v is an endpoint of an interval along which T_e and T_i for some $i < e$ are glued, then there is a half-edge emanating from v along which T_e is not glued to T_i but is glued to some T_j with $j \neq i$. But then v is an endpoint of a (possibly length-zero) interval along which T_i and T_j are glued, hence not a new endpoint. Hence C' contributes an increase of the left-hand side in (1) by at most

$$\#\{\text{boundary points of } C'\} + \sum_{v \text{ interior point of } C'} (2 - l'_v).$$

Straightforward combinatorics shows that this quantity equals 2 for every tree that is not a single point.

Since each component of C contributes at most 2, the left-hand side of (1) increases by at most $2c(C)$ in passing from Γ_{e-1} to Γ_e , and combining this with the first paragraph of the proof we find that the inequality is preserved. \square

Corollary 10. *In the setting of Proposition 9 we have $|E| \leq 2g + 2d - 2$.*

Proof. This follows immediately from the Riemann-Hurwitz condition that $r_\varphi(v) \geq 0$ for each v . \square

We can now prove that the dimension of the locus in $\mathcal{M}_g^{\text{trop}}$ where the gonality is at most d is at most $2g + 2d - 5$.

Proof of Theorem 1, upper bound. Start by taking a gluing datum (T, d, \sim) that gives rise to a degree- d tropical morphism $\varphi : \Gamma \rightarrow T$, and assume that Γ has genus g . If v is a leaf of T such that \sim_v is the partition of $[d]$ into singletons, then $\varphi^{-1}(v)$ consists of d valency-one points. If the interval e leading to v contains points where the gluing is not trivial, then let $w \in e$ be the point closest to v with this property. Otherwise, let $w \in T$ be the point of valency greater than two where e is attached to the rest of T . By deleting the segment $(w, v]$ we obtain a new gluing datum

(T', d, \sim) defining a metric graph Γ' of which Γ is a modification. Proceeding in this manner with deleting unnecessary leaves, we arrive at a gluing datum, which we still denote (T, d, \sim) , such that \sim_v is non-trivial at every leaf $v \in T$. This means that all leaves are endpoints. Next re-attach an interval of positive length at each leaf v where the gluing relation is constant and equal to \sim_v . These intervals do not contribute to the edge lengths of Γ accounted for in the moduli space. We denote the result again by (T, d, \sim) .

Now we count the cardinality $|V \cup E|$ where $V \subseteq T$ is the set of points of valency greater than 2 and E is the set of endpoints of valency at most 2. By basic combinatorics, $|V|$ is at most $l - 2$ where l is the number of leaves, with equality if and only if all points in V are trivalent. Thus, by the corollary above, $|V \cup E| \leq 2g + 2d - 4 + l$. This means that the complement of $V \cup E$ in T consists of at most $2g + 2d - 5 + l$ intervals. Among these intervals, l intervals at the leaves do not contribute to the image of Γ in the moduli space (these intervals may be longer than the ones added in the last modification, if the gluing relation was already constant on a positive-length interval leading into a leaf v). Thus we arrive at the correct dimension count $2g + 2d - 5$.

To describe the locus in $\mathcal{M}_g^{\text{trop}}$ where the gonality equals d one now proceeds as follows:

- (1) Enumerate all (finite, combinatorial) trees with at most $2g + 2d - 5$ edges and where the vertices are allowed to be 2-valent; there are finitely many of these.
- (2) Equip each such tree with a gluing relation \sim that is constant along the edges; again, there are finitely many possibilities.
- (3) Select those combinatorial choices of a tree T plus a gluing that lead to a genus- g graph Γ .
- (4) Assigning positive edge lengths to the edges in T , we arrive at a closed cell of dimension at most $2g + 2d - 5$ in $\mathcal{M}_g^{\text{trop}}$ where the gonality is at most d .

The union of these finitely many cells is the locus in the theorem. \square

4. CONSTRUCTING GRAPHS WITH PRESCRIBED GONALITY

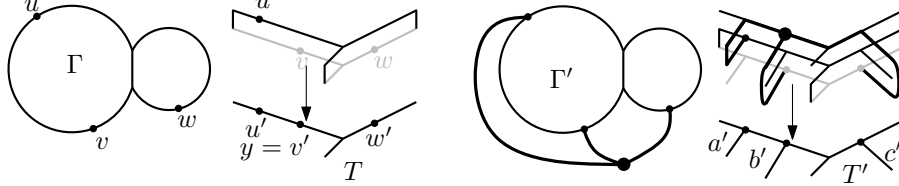
In this section we construct a subset of $\mathcal{M}_g^{\text{trop}}$ of dimension $\min\{2g + 2d - 5, 3g - 3\}$ where the gonality is at most d , thus proving the lower bound in Theorem 1. Consequently, we prove that for $d = \lceil \frac{g}{2} \rceil + 1$ this subset intersects each cell in the definition of $\mathcal{M}_g^{\text{trop}}$ in a non-empty open subset. The main construction is given in the following subsection.

Gluing in a tripod. Let (T, d, \sim) be a gluing datum with corresponding tropical morphism $\varphi : \Gamma \rightarrow T$. Pick points $u, v, w \in \Gamma$ with images $u' := \varphi(u), v' := \varphi(v), w' := \varphi(w) \in T$. These points span a tree in T with at most three leaves; let y be the unique point in T that lies on all the shortest paths $u'v', u'w'$, and $v'w'$. Let a, b, c be the lengths of the shortest paths $u'y, v'y, w'y$, respectively, and pick additional lengths $a', b', c' > 0$. Let $i, j, k \in [d]$ be such that u, v, w lie in T_i, T_j, T_k , respectively. Attach intervals of lengths a', b', c' to T at u', v', w' , respectively, and call the resulting tree T' . Let $u'', v'', w'' \in T'$ be the endpoints of those intervals. Then one obtains a new gluing datum $(T', d + 1, \sim')$ from (T, d, \sim) by gluing T'_{d+1} only along the points u'', v'', w'' with T_i, T_j, T_k , respectively, and leaving the gluing

relation on the other copies of T' unchanged. Let Γ' be the corresponding metric graph. The following is now straightforward.

Lemma 11. *The graph Γ' is a modification of the graph obtained from Γ by adding a new trivalent vertex with new edges attached to u, v, w of lengths $a+2a', b+2b', c+2c'$, respectively.*

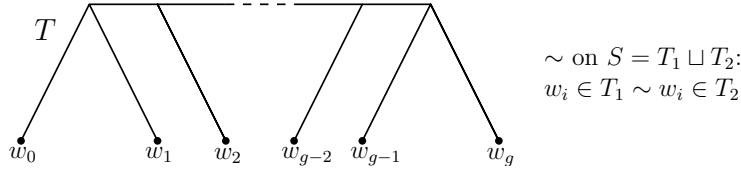
Here is an example with dangling trees in Γ' removed and the top tree attached:



A lower bound on the dimension in Theorem 1.

Proof of Theorem 1, lower bound. We proceed by induction on $d \geq 2$ to exhibit a subset $\mathcal{S}_{g,d}$ of $\mathcal{M}_g^{\text{trop}}$ of dimension $e(g,d) = \min\{2g+2d-5, \dim \mathcal{M}_g^{\text{trop}}\}$ where the gonality is at most d . The subset will be presented by a finite graph $G = (V, E)$ all of whose vertices are trivalent, together with an open polyhedral cone U of dimension e in $\mathbb{R}_{>0}^E$ consisting of edge lengths for which the gonality is at most d .

For $d = 2$ we are concerned with the locus in $\mathcal{M}_g^{\text{trop}}$ of hyperelliptic metric graphs. This locus is well-understood [Cha13], and its dimension is $2g-1 = e(g,2)$ for all $g \geq 1$. Here is a gluing datum witnessing this dimension:



Next let $d \geq 3$ and assume that we have found suitable subsets $\mathcal{S}_{g,d-1}$ for all g . Then to find $\mathcal{S}_{g,d}$ we pick a suitable subset $\mathcal{S}_{g-2,d-1}$, represented by (G, U) . Note that $e(g,d) = e(g-2,d-1) + 6$ provided that $g-2$ is at least 2, since then both of the terms in the minimum increase by 6. Since 6 is also the number of degrees of additional freedom when gluing in a tripod (the positions of u, v, w and the positive numbers a', b', c'), we are done if $g-2 \geq 2$.

We deal with the cases $g = 2, 3$ separately. Every graph with $g = 2$ is hyperelliptic, so for $\mathcal{S}_{g,d}$ with $d \geq 2$ we can take any of the cells defining $\mathcal{M}_2^{\text{trop}}$. This leaves the case where $g = 3$ and $d = 3$. Here $e(3,3) = 6$, and we obtain a subset $\mathcal{S}_{3,3}$ by gluing in a tripod in a cycle—the reason that this raises the dimension by 5 rather than 6 is that the automorphism group of the cycle can move one of the points, say u , to any fixed position. \square

Realising all combinatorial types. We now have almost all ingredients for proving Theorem 2, but one more notion is needed for the existence of a rank-one divisor supported at integral vertices. A finite subset S of a metric graph Γ is called an *integral set* if $\Gamma \setminus S$ is a union of open intervals of length 1 and half-open intervals of length strictly smaller than 1. The metric graph Γ has an integral set if

and only if it is either a line segment of arbitrary length, or a single cycle of integral length, or it has at least one vertex of valency at least three and every line segment connecting two such vertices has integral length. In the last case, the integral set is unique, and the closed ends of the half-open intervals are necessarily valency-one vertices of Γ .

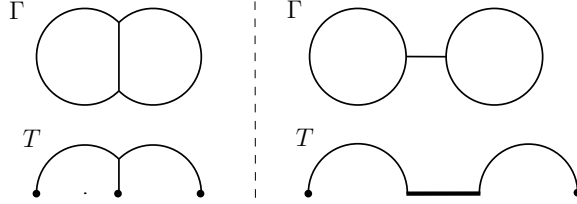
Proof of Theorem 2. We proceed by induction on g to construct an open cone $C_G \subseteq \mathbb{R}_{>0}^E$ for each trivalent graph G of genus g such that graphs Γ represented by points in C_G have gonality at most $\lceil (g+2)/2 \rceil$. Moreover, we will do this in such a way that each metric graph Γ corresponding to a point in $\mathbb{Z}^E \cap C_G$ has a modification Γ' that admits a tropical morphism $\varphi : \Gamma' \rightarrow T$ of degree $\lceil (g+2)/2 \rceil$ to a tree T with the following additional properties:

- (1) φ maps some integral set $S' \subseteq \Gamma'$ into some integral set in T ;
- (2) $S := S' \cap \Gamma$ is an integral set in Γ ; and
- (3) there exists a point $v_0 \in S$ such that $\varphi^{-1}(\varphi(v_0)) \cap \Gamma \subseteq S$.

Then the divisor $\sum_{v' \in \varphi^{-1}(\varphi(v_0))} m_{\varphi}(v')v'$ on Γ' has rank one, and this remains the case if we move the chips on $\Gamma' \setminus \Gamma$ to their nearest point on Γ ; since the points where trees are grafted onto Γ to obtain Γ' are necessarily in the integral set S , this latter divisor is supported on S .

For $g = 1$ it is not quite clear how even to define a graph of genus 1 in which all vertices have valency 3, but we take as definition the *circle* with no vertices. For this graph the statement is clear: for any length $a > 0$ prescribed to the circle, it has a $2 : 1$ -morphism φ to an interval of length $a/2$. If a is integral, then after choosing an integral set S in the cycle, we can choose φ such that $\varphi^{-1}(\varphi(S)) = S$.

For $g = 2$ there are two possible combinatorial types, and the statements to be proved are well known for both (the marked points w on T have equivalence class $\{1, 2\}$ for \sim_w):



In each of the two cases, the open cone equals $\mathbb{R}_{>0}^E$, and also the integrality statements are readily verified.

Next, assume that $G = (V, E)$ is a trivalent graph of genus $g > 2$. If G has a vertex y that can be removed without disconnecting the graph, then let $G' = (V', E')$ be the graph obtained by removing y and its three incident edges e_1, e_2, e_3 . Note that G' has genus $g - 2$. By assumption, there is an open cone $C_{G'} \subseteq \mathbb{R}_{>0}^{E'}$ of dimension $\dim \mathcal{M}_{g-2}^{\text{trop}}$ consisting of edge lengths leading to metric graphs with gonality at most $\lceil \frac{g}{2} \rceil$. By gluing in a tripod we find an open cone $C_G \subseteq \mathbb{R}^E$ of the right dimension where the gonality is at most $1 + \lceil \frac{g}{2} \rceil = \lceil \frac{g+2}{2} \rceil$: the inequalities for C_G are those for $C_{G'}$ plus the conditions that the lengths of e_1, e_2, e_3 are sufficiently large (in the terminology of the subsection on gluing a tripod: larger than a, b, c , respectively).

To see that the integrality conditions are preserved, let Γ be the metric graph corresponding to a point in $C_G \cap \mathbb{Z}^E$. The restriction to E' defines a metric graph Σ of combinatorial type G' , and by the induction hypothesis there is tropical morphism ψ from a modification Σ' of Σ to a tree K of degree $\lceil \frac{g}{2} \rceil$ that satisfies the integrality conditions: Σ' has an integral set R' such that $\psi(R')$ is contained in an integral set U of K , $R := \Sigma \cap R'$ is an integral set in Σ , and $v_0 \in R$ is such that $\psi^{-1}(\psi(v_0)) \cap \Sigma \subseteq R$.

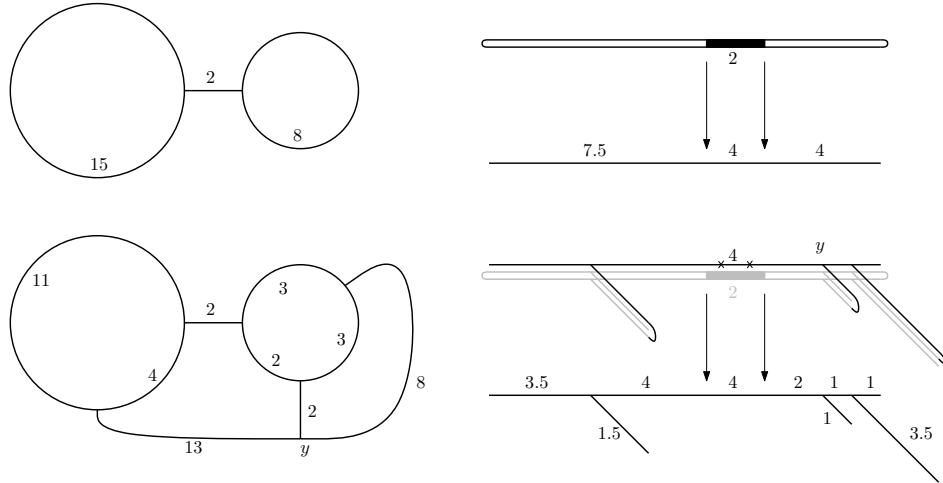
Let φ be the tropical morphism $\Gamma' \rightarrow T$ obtained via gluing a tripod to the endpoints $u, v, w \in \Sigma$ of e_1, e_2, e_3 with edge lengths $a + 2a', b + 2b', c + 2c'$ as in the subsection on tripods, and with central vertex y . Here Γ' is a modification of Γ . Then $u, v, w \in R^1$, and hence $\psi(u), \psi(v), \psi(w) \in U$. We extend U to an integral set V of T by adding the vertices on the new edges (of lengths $a', b', c' \in \frac{1}{2}\mathbb{Z}$) at an integral distance from $\psi(u), \psi(v), \psi(w)$, respectively. Next, we extend R' to an integral set S' of Γ' by

$$S' := R' \cup \{v \in \Gamma' \setminus \Sigma' \mid \varphi(v) \in V\}.$$

Set $S := S' \cap \Gamma$. Then we find that each $x \in \varphi^{-1}(\varphi(v_0)) \cap \Gamma$ is either in $\psi^{-1}(\psi(v_0)) \cap \Sigma \subseteq R \subseteq S$, or else in $\Gamma' \setminus \Sigma'$ and hence, since $\varphi(x) = \varphi(v_0) \in U$, also $x \in S$. Thus S' has the required property.

This concludes the proof for the case where G has a trivalent vertex y such that removing y does not disconnect the graph. If, on the other hand, removing *any* vertex y of G disconnects G , then any two distinct simple cycles in G intersect in at most one vertex, i.e., G is a *cactus graph*. This case is dealt with by the proposition below. \square

Example 12. In the following example, we see two metric graphs of genera 2 and 4, respectively, on the left, along with tropical morphisms of degrees 2 and 3, as constructed above:



Focussing on the latter morphism φ , note that the segment between the two arrows has length 4 in the tree, and also in one of the copies of the tree above, but length

¹Actually, there is one case where this is not automatic, namely, when Σ is a single cycle. But in this case we can take Σ' equal to Σ and *choose* R' such that it contains u, v, w .

2 where the other two copies are glued together. The two marked points are in the graph's integral set, but have a strictly half-integral point in their fibre. So they would *not* be a valid choice for v_0 in condition (3) above.

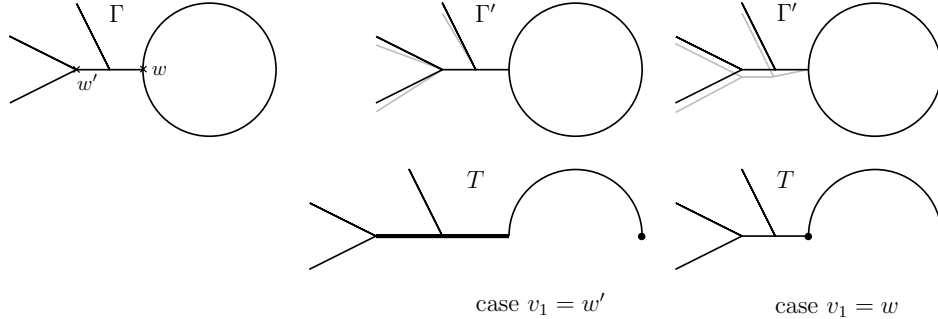
Cactus graphs. A metric graph Γ is called a *cactus graph* if any two simple cycles (i.e., injective, continuous images of S^1) intersect in at most one point.

Proposition 13. *Any metric cactus graph Γ has a modification Γ' with a tropical morphism φ from Γ' to a tree T , of degree $\lceil \frac{g(\Gamma)+2}{2} \rceil$, with the following additional constraints:*

- (1) *if $g(\Gamma)$ is odd and $v_1 \in \Gamma$ is any point, then φ can be chosen such that $m_\varphi(v_1) = 2$ and that moreover $k \geq 2l - 1$ where k is the valency of v_1 in Γ and l is the valency of $\varphi(v_1)$ in T ;*
- (2) *if Γ has an integral set S (containing v_1 if $g(\Gamma)$ is odd), then Γ' has an integral set S' containing S , $\varphi(S')$ is contained in an integral set of T , and $\varphi^{-1}(\varphi(S')) \subseteq S'$.*

The former condition implies that the Riemann-Hurwitz inequality $k - 2 \geq 2(l - 2)$ holds at v_1 with a strict inequality. The latter condition is stronger than condition (3) in the proof above, where we require only that the fibre through *some* integral point intersects Γ only inside S . Example 12 below shows why we could not impose this stronger condition earlier.

Proof. We proceed by induction on g . For $g = 0$ we take $T = \Gamma' = \Gamma$ and φ the identity map. For $g = 1$ let C be the unique simple cycle in Γ , of length $a > 0$, and let w be the point on C closest to the prescribed point v_1 . The $2 : 1$ map from C to an edge with branchpoints w and the point at distance $a/2$ from w extends to a modification of Γ that has slope 1 everywhere except for slope 2 on the segment connecting v_1 and w :



For (1) we note that k is at least $2l - 1$, as required. For the integrality condition (2), we note that any integral set S of Γ has a unique extension to an integral set S' of Γ' , and that there is a unique integral set in T containing the image of S' . Note that the latter inclusion is strict if the segment from v_1 to w has positive (integral) length. Yet, $\varphi^{-1}(\varphi(S)) = S$, as required.

If Γ has higher genus, then we can write it as $\Gamma = \Gamma_1 \cup \Gamma_2$ where $\Gamma_1, \Gamma_2 \subseteq \Gamma$ are cactus graphs of lower genus than Γ that intersect in a single point y of Γ , which we can chose integral if Γ has an integral set. Write $g := g(\Gamma)$ and $g_i := g(\Gamma_i)$ for $i = 1, 2$, and note that $g = g_1 + g_2$. Moreover, if g is odd, then we can (and do)

choose the decomposition such that g_1 is odd and that the prescribed point v_1 lies in Γ_1 . Furthermore, if Γ has an integral set S , then $S \cap \Gamma_i$ is an integral set for Γ_i for each $i = 1, 2$.

By the induction hypothesis, there are modifications Γ'_i , $i = 1, 2$ of the Γ_i with tropical morphisms φ_i of degrees $\lceil g_i/2 \rceil + 1$ to trees T_i which further satisfy conditions (1) and (2) in the proposition. Here we choose v_1 equal to y for both φ_1 and φ_2 if both g_1 and g_2 are odd (in this case, since g is even, no point v_1 had yet been prescribed).

Let T be the tree obtained by gluing T_1 and T_2 at $\varphi_1(y)$ and $\varphi_2(y)$, respectively, and let Γ' be the metric graph obtained by gluing Γ'_1, Γ'_2 at y . The metric graph Γ' is a modification of Γ , and an integral set of Γ extends uniquely to one of Γ' . Let $\psi : \Gamma' \rightarrow T$ be the map restricting to φ_i on Γ'_i . Then ψ is harmonic except in the points in $Y := \psi^{-1}(\psi(y))$. Apart from y , which belongs to both Γ'_i , the points in Y are either in Γ'_1 or in Γ'_2 but not both. Let $v_0 := y, v_1, \dots, v_a$ be the points in $Y \cap \Gamma'_1$ and let $w_0 := y, w_1, \dots, w_b$ be the points in $Y \cap \Gamma'_2$, and write $m_i := m_{\varphi_1}(v_i)$ for $i = 0, \dots, a$ and $n_j := m_{\varphi_2}(w_j)$ for $j = 0, \dots, b$. We modify Γ' by grafting m_i copies of T_2 at v_i for $i = 1, \dots, a$ (not yet at $v_0 = y$!) and grafting n_j copies of T_1 at w_j for $j = 1, \dots, b$ (not yet at $w_0 = y$!), and we extend ψ to these copies by their natural maps into T . This renders ψ harmonic at v_1, \dots, v_a and w_1, \dots, w_b , and moreover restores the Riemann-Hurwitz condition there—e.g., to the valency v_i one adds m_i times the valency of $\varphi_2(y)$ in T_2 , which is exactly m_i times what was added to the valency of $\varphi_1(v_i) = \varphi(y)$ by attaching T_2 .

So we need only establish harmonicity and the Riemann-Hurwitz condition at y . Let $d_i = \lceil g_i/2 \rceil + 1$ be the degree of φ_i . First, assume that g_2 is even. Then the required degree of the map φ equals

$$d := \lceil g/2 \rceil + 1 = \lceil g_1/2 \rceil + g_2/2 + 1 = d_1 + d_2 - 1.$$

We graft $m_0 - 1$ further copies of T_2 and $n_0 - 1$ copies of T_1 to y ; this yields the final modification Γ'' of Γ with the natural extension $\varphi : \Gamma'' \rightarrow T$ of ψ . This extension is harmonic everywhere by construction. To check the Riemann-Hurwitz condition at y , let k_i, k denote the valency of y in Γ'_i and Γ'' , respectively, and let l_i, l denote the valency of $\varphi_i(y)$ and $\varphi(y)$ in T_i, T , respectively. Then we have $l = l_1 + l_2$ and $k = k_1 + k_2 + (m_0 - 1)l_2 + (n_0 - 1)l_1$, so that

$$\begin{aligned} k - 2 &= k_1 + k_2 + (m_0 - 1)l_2 + (n_0 - 1)l_1 - 2 \\ &\geq m_0(l_1 - 2) + 2 + n_0(l_2 - 2) + 2 + (m_0 - 1)l_2 + (n_0 - 1)l_1 - 2 \\ &= (m_0 + n_0 - 1)(l_1 + l_2 - 2) \\ &= m_\varphi(y)(l - 2), \end{aligned}$$

where the inequality follows from the Riemann-Hurwitz inequalities for the φ_i .

Second, assume that g_2 is odd; then, by assumption, so is g_1 . With notation as above we now have

$$d = \lceil g/2 \rceil + 1 = d_1 + d_2 - 2.$$

Moreover, since we had chosen y as the prescribed point for both φ_1 and φ_2 , we have $m_0 = n_0 = 2$ by property (1). This means that we need not graft further trees at y and the map $\varphi : \Gamma'' \rightarrow T$ constructed so far is harmonic there. To check the Riemann-Hurwitz conditions, we compute

$$k - 2 = k_1 + k_2 - 2 \geq 2l_1 - 1 + 2l_2 - 1 - 2 = 2(l - 2) = m_\varphi(y)(l - 2),$$

where we have used that $k_i \geq 2l_i - 1$ by property (1).

Finally, if Γ has an integral set S , then Γ'' has a unique integral set S' containing S , and it contains the integral sets $S'_i = S' \cap \Gamma'_i$ as well as suitable integral sets of the trees grafted onto Γ' to arrive at Γ'' . The points in $\varphi^{-1}(\varphi(S'))$ that are not in the union of the sets $\varphi_i^{-1}(\varphi_i(S'_i))$ are in those integral sets of the grafted trees, hence also in S' . \square

- Remark 14.** (1) Combining this subsection with the previous one, we find that metric graphs in the open cone C_G have a modification with a tropical morphism to a tree that only has slopes 1 and 2.
- (2) The existence of divisors of *higher* rank on cactus graphs, under the condition that the Brill-Noether number is nonnegative, was studied by Jorn van der Pol in his Bachelor's thesis [vdP11] under an additional assumption on the cactus graph.
- (3) The existence of integral rank-one divisors of degree $\lceil (g+2)/2 \rceil$ on an arbitrary graph of genus g remains conjectural. Conceivably, Backman's approach to linear equivalence using graph orientations [Bac14] could lead to such a result. In any case, following our approach in this paper, we do not see how to cross the boundary of the cone C_G .

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